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A GENERALIZATION OF ULTRASPHERICAL POLYNOMIALS.(U)

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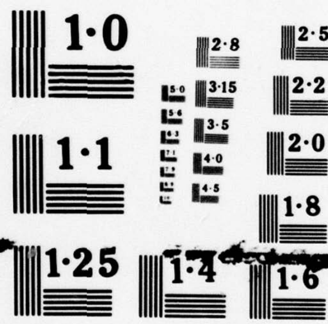

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A GENERALIZATION OF ULTRASPHERICAL POLYNOMIALS

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ABSTRACT

Some old polynomials of L. J. Rogers are orthogonal. Their weight function is given. The connection coefficient problem, which Rogers solved by guessing the formula and proving it by induction, is derived in a natural way and some other formulas are obtained. These polynomials generalize zonal spherical harmonics on spheres and include as special cases polynomials that are spherical functions on rank one spaces over reductive p-adic groups. A limiting case contains some Jacobi polynomials studied by Hylleraas that arose in work on the Yukawa potential.

AMS (MOS) Subject Classifications: 10A45, 33A30, 33A45, 33A65

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SIGNIFICANCE AND EXPLANATION

Spherical harmonics are used to solve many physical problems, especially in potential theory. A generalization of zonal spherical harmonics was introduced by L. J. Rogers in 1895. He obtained many properties of these polynomials, including some that would not be found for the classical spherical harmonics for another twenty-five years. However he was unaware that his polynomials were orthogonal. The orthogonality relation is proved and used to obtain further results for these polynomials. Another limiting case gives a relatively recent result of Hylleraas that arose in his study of the Yukawa potential.

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A GENERALIZATION OF ULTRASPHERICAL POLYNOMIALS

Richard Askey and Mourad E.-H. Ismail

To the memory of Paul Turán, with respect and affection.

1. Introduction. Fejér [21] introduced the following class of polynomials. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

be a function analytic in a neighborhood of the origin with a_n real. Form

$$|f(re^{i\theta})|^2 = \sum_{n=0}^{\infty} P_n(\cos\theta) r^n , \tag{1.2}$$

so that

$$P_n(\cos\theta) = \sum_{k=0}^{n} a_k a_{n-k} \cos(n-2k)\theta . \tag{1.3}$$

$P_n(x)$ is a polynomial of degree n, and is called a generalized Legendre polynomial since $f(z) = (1-z)^{-1/2}$ gives the Legendre polynomials. More generally, $f(z) = (1-z)^{-\lambda}$ gives the ultraspherical polynomials $C_n^\lambda(x)$. The orthogonality relation for the ultraspherical polynomials when $\lambda > -\frac{1}{2}$ is

$$\begin{aligned}\int_{-1}^{1} C_n^\lambda(x) C_m^\lambda(x) (1-x^2)^{\lambda-1/2} dx &= 0, \quad m \neq n , \\ &= \frac{(2\lambda)_n \Gamma(\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}{n!(n+\lambda)\Gamma(\lambda)}, \quad m = n .\end{aligned} \tag{1.4}$$

Fejér and Szegö obtained a number of interesting facts about these generalized Legendre polynomials, some of which are summarized in [40, Chapter VI]. Feldheim [23] and Lancevickiĭ [32] determined when the generalized Legendre polynomials are orthogonal by showing that the polynomials must satisfy a specific recurrence relation. However they did not obtain an explicit representation for the polynomials and they were unable to find the weight function. We will find both the polynomials and the weight

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function. These polynomials are not new. They were studied extensively by Rogers in the third of an important series of papers [36]. The only results from this series of papers that are well known are two identities that Ramanujan rediscovered and are now known as the Rogers-Ramanujan identities. However there are many other interesting identities (some are reproved in [6]), and the polynomials in the third paper are probably more interesting and important than any other results in Rogers' papers, including the Rogers-Ramanujan identities. We will summarize some of the results Rogers obtained for these polynomials and add a few new ones.

The key to obtaining explicit formulas is to use the q-binomial theorem and basic hypergeometric series. The binomial theorem can be written as

$$(1 - x)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n \tag{1.5}$$

where

$$(a)_n = \Gamma(n + a)/\Gamma(a) = a(a + 1) \cdots (a + n - 1) . \tag{1.6}$$

The q-binomial theorem [5, Th. 2.1] is

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1, \quad |q| < 1 , \tag{1.7}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1 , \tag{1.8}$$

and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) . \tag{1.9}$$

More general hypergeometric and basic hypergeometric series are given by

$${}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{x^n}{n!} \tag{1.10}$$

and

$${}_{r+1}\varphi_r\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_{r+1};q)_n}{(b_1;q)_n \cdots (b_r;q)_n} \frac{x^n}{(q;q)_n} . \tag{1.11}$$

To motivate the study of the polynomials implicitly found by Feldheim and Lancevickiĭ we remark that their generating function will turn out to be

$$\frac{(\beta re^{i\theta};q)_\infty(\beta re^{-i\theta};q)_\infty}{(\alpha re^{i\theta};q)_\infty(\alpha re^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty} P_n(\cos\theta)r^n , \tag{1.12}$$

which is a very natural analogue of the classical generating function

$$(1 - 2r\cos\theta + r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(\cos\theta)r^n . \tag{1.13}$$

For $1 - 2r\cos\theta + r^2 = (1 - re^{i\theta})(1 - re^{-i\theta})$, and the analogy between the binomial theorem as given in (1.5) and the q-binomial theorem in (1.7) shows that the function $(\beta re^{i\theta};q)_\infty/(\alpha re^{i\theta};q)_\infty$ is a natural substitute for $(1 - re^{i\theta})^{-\lambda}$.

2. The orthogonal generalized Legendre polynomials. If a set of polynomials is orthogonal and satisfies $P_n(-x) = (-1)^n P_n(x)$, then it must satisfy the three term recurrence relation

(2.1) $\quad 2b_n x P_n(x) = P_{n+1}(x) + \lambda_n P_{n-1}(x), \quad n = 0,1,\ldots, \quad P_0(x) = 1, \quad P_{-1}(x) = 0 .$

Feldheim [23] determined b_n and λ_n for the generalized Legendre polynomials that are orthogonal. His result is

$$b_n = b_1 + (b_1 - b_0)\frac{\sinh(n-1)\xi}{\sinh(n+1)\xi}, \quad n = 0,1,\ldots,$$

$$\lambda_n = b_1^2 + 2b_1(b_1 - b_0)\frac{\sinh(n-1)\xi}{\sinh(n+1)\xi} + (b_1 - b_0)^2\frac{\sinh(n-3)\xi}{\sinh(n+1)\xi} .$$

The coefficients in the original power series for $f(z)$ satisfy

(2.2) $$b_n = a_n/a_{n-1} .$$

Setting $q = \exp(-2\xi)$, we rewrite these as

(2.3) $$b_n = b_1 + (b_1 - b_0)q\,\frac{(1 - q^{n-1})}{(1 - q^{n+1})}$$

(2.4) $$\lambda_n = b_1^2 + 2b_1(b_1 - b_0)q\,\frac{(1 - q^{n-1})}{(1 - q^{n+1})} + (b_1 - b_0)^2 q^2\,\frac{(1 - q^{n-3})}{(1 - q^{n+1})} .$$

With these choices of b_n and λ_n, formula (2.1) can be written as

$$\begin{aligned} &2x[b_1(1 + q) - b_0 q - (b_1 q + b_1 - b_0)q^n]P_n(x) \\ &\quad = (1 - q^{n+1})P_{n+1}(x) + [(b_1 + b_1 q - b_0 q)^2 - q^{n-1}(b_1 + b_1 q - b_0)^2]P_{n-1}(x) . \end{aligned}$$

Set $b_1 + b_1 q - b_0 q = \alpha$ and $b_1 q + b_1 - b_0 = \beta$. This gives

(2.5) $$2x[\alpha - \beta q^n]P_n(x) = (1 - q^{n+1})P_{n+1}(x) + [\alpha^2 - \beta^2 q^{n-1}]P_{n-1}(x) .$$

To find the polynomials $P_n(x)$, multiply (2.5) by r^{n+1} and sum, recalling that $P_{-1}(x) = 0$, and $P_0(x) = 1$. If

$$f(r,x) = \sum_{n=0}^{\infty} P_n(x) r^n ,$$

the resulting equation is

$$2x\alpha r f(r,x) - 2x\beta r f(qr,x) = f(r,x) - 1 - [f(qr,x) - 1] + \alpha^2 r^2 f(r,x) - \beta^2 r^2 f(qr,x)$$

or

$$(1 - 2x\alpha r + \alpha^2 r^2) f(r,x) = (1 - 2x\beta r + \beta^2 r^2) f(qr,x) \; . \tag{2.6}$$

Set $x = \cos\theta$ and rewrite (2.6) as

$$f(r,\cos\theta) = \frac{(1 - \beta re^{i\theta})(1 - \beta re^{-i\theta})}{(1 - \alpha re^{i\theta})(1 - \alpha re^{-i\theta})} f(qr,\cos\theta) \; . \tag{2.7}$$

Iterate (2.7) to obtain

$$f(r,\cos\theta) = \frac{(\beta re^{i\theta};q)_n (\beta re^{-i\theta};q)_n}{(\alpha re^{i\theta};q)_n (\alpha re^{-i\theta};q)_n} f(q^n r,\cos\theta) \; . \tag{2.8}$$

Then let $n \to \infty$. The result is

$$f(r,\cos\theta) = \frac{(\beta re^{i\theta};q)_\infty (\beta re^{-i\theta};q)_\infty}{(\alpha re^{i\theta};q)_\infty (\alpha re^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty} P_n(\cos\theta) r^n \; . \tag{2.9}$$

This is the generating function stated in (1.12).

The above derivation was formal, but it is easy to justify it, since the function $f(z,\cos\theta)$ is analytic for $|z| < |\alpha|^{-1}$. There is no loss in generality in taking $\alpha = 1$, for it can be removed by scaling. With this simplification define $C_n(x;\beta|q)$ by

$$\sum_{n=0}^{\infty} C_n(x;\beta|q) r^n = \frac{(\beta e^{i\theta}r;q)_\infty (\beta e^{-i\theta}r;q)_\infty}{(e^{i\theta}r;q)_\infty (e^{-i\theta}r;q)_\infty} \; . \tag{2.10}$$

We will use x and $\cos\theta$ interchangeably. When $\beta = q^\lambda$ it is easy to see that

$$\lim_{q\to 1^-} C_n(x;q^\lambda|q) = C_n^\lambda(x) \; . \tag{2.11}$$

This follows from either the generating function (2.10), using

$$\lim_{q\to 1^-} \frac{(q^\lambda x;q)_\infty}{(x;q)_\infty} = \lim_{q\to 1^-} \sum_{n=0}^{\infty} \frac{(q^\lambda;q)_n}{(q;q)_n} x^n = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} x^n = (1 - x)^{-\lambda} \; , \tag{2.12}$$

or from the recurrence relation

$$2x[1 - \beta q^n] C_n(x;\beta|q) = (1 - q^{n+1}) C_{n+1}(x;\beta|q) + (1 - \beta^2 q^{n-1}) C_{n-1}(x;\beta|q) \tag{2.13}$$

which becomes

$$(2.14) \qquad 2x(n+\lambda)C_n^\lambda(x) = (n+1)C_{n+1}^\lambda(x) + (n+2\lambda-1)C_{n-1}^\lambda(x)$$

when $\beta = q^\lambda$, every term is divided by $1 - q$ and $q \to 1^-$.

If $C_n(x;\beta|q) = k_n r_n(x;\beta|q)$ with $k_n = (\beta;q)_n/(q,q)_n$, then (2.13) becomes

$$(2.15) \qquad 2xr_n(x) = r_{n+1}(x) + \frac{(1-\beta^2 q^{n-1})(1-q^n)}{(1-\beta q^{n-1})(1-\beta q^n)} r_{n-1}(x) .$$

Now we see that the case $|q| > 1$ can be reduced to the case $|q| < 1$ when β is replaced by β^{-1}. Thus there was no loss in generality in assuming $|q| \leq 1$. We use β rather than q^λ since β could be negative, and the case $\beta = 0$ is very important. The conditions for orthogonality can be obtained from (2.15). See [20], [27, Ch. II, Th. 1.5]. It is necessary and sufficient that

$$\frac{(1-\beta^2 q^{n-1})(1-q^n)}{(1-\beta q^{n-1})(1-\beta q^n)} > 0, \qquad n = 1,2,\ldots .$$

When $0 < q < 1$, the cases $n = 1$ and $n = 2$ imply $-1 < \beta < q^{-1/2}$, and these conditions are easily seen to be sufficient. When $-1 < q < 0$ the cases $n = 1$, $n = 2$, and $n = 3$ imply $-1 < \beta < -q^{-1}$ and these conditions are sufficient. The trivial case $q = 0$ holds when $\beta > -1$. The only way this inequality can hold when $q \to 1$ is to have $\beta = q^\lambda$, in which case the condition is $\lambda > -\frac{1}{2}$. Finally the case $q \to -1$ can only hold when $\beta = |q|^\lambda$ and $\lambda > -1$ or $\beta = -|q|^\lambda$ and $\lambda > 0$.

3. Explicit representations for the continuous q-ultraspherical polynomials. There is another set of orthogonal polynomials that are basic hypergeometric series and have the ultraspherical polynomials as limits when $q \to 1$. These will be called the discrete q-ultraspherical polynomials, because their distribution function is a discrete measure. See [7]. The weight functions for the polynomials under consideration in this paper are absolutely continuous when $|\beta| < 1$, so these polynomials will be called the continuous q-ultraspherical polynomials. The adjective will be used in both cases, since it is not clear which will be more important.

To find an explicit representation use the q-binomial theorem (1.7) in the generating function (2.10). The result is $(x = \cos\theta)$

$$C_n(x;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} e^{i(n-2k)\theta} \tag{3.1}$$

$$= \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} \cos(n-2k)\theta$$

$$= \frac{(\beta;q)_n}{(q;q)_n} e^{in\theta} \sum_{k=0}^{n} \frac{(q^{-n};q)_k(\beta;q)_k}{(q^{1-n}\beta^{-1};q)_k(q;q)_k} (q\beta^{-1}e^{-2i\theta})^k$$

since $(a;q)_{n-k} = \dfrac{(a;q)_n q^{k(k+1)/2}}{(-aq^n)^k(q^{1-n}a^{-1};q)_k}$. This gives

$$C_n(\cos\theta;\beta|q) = \frac{(\beta;q)_n}{(q;q)_n} e^{in\theta} {}_2\varphi_1\left[\begin{matrix} q^{-n},\beta \\ q^{1-n}\beta^{-1} \end{matrix}; q, q\beta^{-1}e^{-2i\theta}\right]. \tag{3.2}$$

Another representation can be obtained as a special case of a more general set of orthogonal polynomials considered in [12]. The polynomials are

$$P_n(x) = {}_4\varphi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q,q\right] \tag{3.3}$$

$x = \cos\theta$. Their recurrence relation is

$$2xP_n(x) = a_nP_{n+1}(x) - (a_n + c_n - a - a^{-1})P_n(x) + c_nP_{n-1}(x) , \tag{3.4}$$

where

$$(3.5)\qquad a_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}$$

$$(3.6)\qquad c_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})} .$$

First set $c = -a$, $d = -b$. In this case the weight function is even and $a_n + c_n = a + a^{-1}$. Then set $b = aq^{1/2}$. With these specializations, the coefficients a_n and c_n are

$$(3.7)\qquad a_n = \frac{(1-a^4q^n)}{a(1-a^2q^n)}$$

$$(3.8)\qquad c_n = \frac{a(1-q^n)}{(1-a^2q^n)} .$$

Set $P_n(x) = k_n s_n(x)$ with $k_n = a^n(a^2;q)_n/(a^4;q)_n$. The recurrence relation (3.4) becomes

$$(3.9)\qquad 2xs_n(x) = s_{n+1}(x) + \frac{(1-a^4q^{n-1})(1-q^n)}{(1-a^2q^{n-1})(1-a^2q^n)} s_{n-1}(x) .$$

This gives $r_n(x) = s_n(x)$ with $\beta = a^2$, where $r_n(x)$ satisfies (2.15). Combining the above formulas we see that

$$(3.10)\qquad C_n(\cos\theta;\beta|q) = \frac{(\beta^2;q)_n}{\beta^{n/2}(q;q)_n}\, {}_4\varphi_3\left[\begin{matrix} q^{-n},\ q^n\beta^2,\ \beta^{1/2}e^{i\theta},\ \beta^{1/2}e^{-i\theta} \\ \beta q^{1/2},\ -\beta q^{1/2},\ -\beta \end{matrix} ;\ q,q\right] .$$

Another way of writing (3.10) is

$$(3.11)\qquad C_n(x;\beta|q) = \frac{(\beta^2;q)_n}{\beta^{n/2}(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^n\beta^2;q)_k \prod_{j=0}^{k-1}(1-2\beta^{1/2}xq^j+\beta q^{2j})}{(\beta q^{1/2},q)_k(-\beta q^{1/2};q)_k(-\beta;q)_k(q;q)_k} q^k .$$

This is probably as close as one can get to a single sum that gives these polynomials as a series in the polynomial variable x. Another formula which shows the polynomial character of $C_n(x;\beta|q)$ is

$$(3.12)\qquad C_n(x;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} T_{n-2k}(x) ,$$

where $T_n(x)$ is the Chebychev polynomial of the first kind defined by $T_n(\cos\theta) = \cos n\theta$.

Two interesting formulas follow from (3.12). One is the special case $\beta = q$.

$$C_n(\cos\theta; q|q) = \sum_{k=0}^{n} \cos(n-2k)\theta = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{3.13}$$

The second is

$$\lim_{\beta\to 1} (1-q^n) \frac{C_n(\cos\theta;\beta|q)}{2(1-\beta)} = \cos n\theta, \quad n = 1,2,\ldots . \tag{3.14}$$

Observe that in both cases the polynomials are independent of q.

In the case of ultraspherical polynomials the zeros of the polynomials $C_n^\lambda(x)$ have absolute values that decrease as λ increases. This suggests that the zeros of $C_n(\cos\theta;q^\lambda|q)$ lie between the zeros of $\cos n\theta$ and $\sin(n+1)\theta$ when $0<\lambda<1$ and $0<q<1$. Feldheim [23] has shown this, but his notation is so much at variance with ours that we give a new proof. By a theorem of Fejér [22], [40, Theorem 6.5.2] it is sufficient to show that the coefficients are a moment sequence on [0,1], i.e.

$$a_n = \int_0^1 t^n d\mu(t), \; d\mu(t) \geqq 0 .$$

Thus we need

$$\frac{(q^\lambda;q)_n}{(q;q)_n} = \int_0^1 t^n d\mu(t) . \tag{3.15}$$

Let $d\mu$ have a point mass equal to

$$\frac{(q^\lambda;q)_\infty (q^{1-\lambda};q)_\infty}{(q;q)_\infty^2} \frac{t^{n+\lambda}(tq;q)_\infty}{(tq^{1-\lambda};q)_\infty} \quad \text{at} \quad t = q^i, \qquad i = 0,1,\ldots .$$

When $0<\lambda<1$ this mass is positive, and the q-binomial theorem (1.7) shows that (3.15) holds. The inequalities for the zeros are

$$\begin{aligned} \frac{\pi(k-\frac{1}{2})}{n} &\leqq \theta_{k,n}^\lambda \leqq \frac{k\pi}{n+1}, \quad 0<\lambda<1,\ 0<q<1,\ k = 1,\ldots,\left[\frac{n}{2}\right], \\ \frac{k\pi}{n+1} &\leqq \theta_{k,n}^\lambda \leqq \frac{\pi(k-\frac{1}{2})}{n}, \quad 0<\lambda<1,\ 0<q<1,\ l = \left[\frac{n}{2}\right],\ldots,n , \end{aligned} \tag{3.16}$$

where $\theta_{k,n}^\lambda$ are the zeros of $C_n(\cos\theta;q^\lambda|q)$ ordered by $\theta_{k,n}^\lambda < \theta_{k+1,n}^\lambda$. There is equality in (3.16) only when n is odd and $k = (n+1)/2$.

An unlikely looking formula follows on equating the basic hypergeometric series in (3.2) and (3.10). It can be written as

$$(3.17)\qquad {}_2\varphi_1\left[\begin{matrix} q^{-n},\ a^2 \\ q^{1-n}a^{-2} \end{matrix};\ q,\ \frac{qx^2}{a^2}\right] = \frac{x^n(a^4;q)_n}{a^n(a^2;q)_n}\ {}_4\varphi_3\left[\begin{matrix} q^{-n},\ q^na^4,\ ax,\ ax^{-1} \\ a^2q^{1/2},\ -a^2q^{1/2},\ -a^2 \end{matrix};\ q,q\right] .$$

When $a = q^{\alpha/2}$ and the limit $q \to 1^-$ is taken the resulting identity is

$$(3.18)\qquad {}_2F_1\left[\begin{matrix} -n,\ \alpha \\ 1-n-\alpha \end{matrix};\ x^2\right] = \frac{x^n(2\alpha)_n}{(\alpha)_n}\ {}_2F_1\left[\begin{matrix} -n,\ n+2\alpha \\ \alpha+\frac{1}{2} \end{matrix};\ -\frac{(1-x)^2}{4x}\right] .$$

This is one of the iterated quadratic transformations. The first quadratic type transformation for basic hypergeometric series seems to be that of Carlitz [17]. Another is given in [7], where the discrete q-ultraspherical polynomials are related to some discrete q-Jacobi polynomials. Another will be given in [12].

An interesting inequality follows from (3.1),

$$(3.19)\qquad |C_n(x;\beta|q)| \leqq C_n(1;\beta|q),\quad -1 < \beta < 1,\ -1 < q < 1 .$$

For $|\cos(n-2k)\theta| \leqq 1$ and $(\beta;q)_k > 0,\ (q;q)_k > 0$ when $-1 < \beta < 1,\ -1 < q < 1$.

Unlike the classical case of $C_n^\lambda(x)$, when

$$(3.20)\qquad C_n^\lambda(1) = \frac{(2\lambda)_n}{n!} ,$$

it is impossible to find the value $C_n(1;\beta|q)$ as a simple product. There are two interesting points where the value can be given as a product. From (2.10)

$$\sum_{n=0}^\infty C_n(0;\beta|q)r^n = \frac{(\beta ir;q)_\infty(-\beta ir;q)_\infty}{(ir;q)_\infty(-ir;q)_\infty} = \frac{(-\beta^2r^2;q^2)_\infty}{(-r^2;q^2)_\infty} = \sum_{n=0}^\infty \frac{(\beta^2;q^2)_n}{(q^2;q^2)_n}(-1)^n r^{2n}$$

so

$$(3.21)\qquad C_{2n}(0;\beta|q) = (-1)^n\frac{(\beta^2;q^2)_n}{(q^2;q^2)_n}$$

$$C_{2n+1}(0;\beta|q) = 0 .$$

Heine [28], see also [5, Cor. 2.4] found an analogue of Gauss' sum of ${}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix};\ 1\right]$.

It is

(3.22)
$$ {}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix} ; q, \frac{c}{ab}\right) = \frac{\left(\frac{c}{a};q\right)_\infty \left(\frac{c}{b};q\right)_\infty}{\left(\frac{c}{ab};q\right)_\infty (c;q)_\infty}, \quad \left|\frac{c}{ab}\right| < 1, \ |q| < 1 . $$

If the series terminates this is the correct value without the condition $\left|\frac{c}{ab}\right| < 1$.

Set $e^{2i\theta} = \beta$ in (3.2). This gives

(3.23)
$$ C_n\left(\frac{\beta^{1/2}+\beta^{-1/2}}{2};\beta|q\right) = \frac{(\beta^2;q)_n}{(q;q)_n}\beta^{-n/2}, \quad 0 < \beta < q^{-1/2} . $$

When $1 < \beta < q^{-1/2}$ this is interesting, for then

$$ \lim_{n\to\infty} C_n\left(\frac{\beta^{1/2}+\beta^{-1/2}}{2};\beta|q\right) = 0 . $$

This goes to zero fast enough so the distribution function has a point mass at $x = (\beta^{1/2}+\beta^{-1/2})/2$ when $1 < \beta < q^{-1/2}$, see [33]. By symmetry there is also one at $x = -(\beta^{1/2}+\beta^{-1/2})/2$. These are the only point masses when $1 < \beta < q^{-1/2}$.

For convenience the first four of these polynomials are given next.

$$ C_0(x;\beta|q) = 1 $$

$$ C_1(x;\beta|q) = \frac{2(1-\beta)}{(1-q)}x $$

(3.24)
$$ C_2(x;\beta|q) = \frac{4(1-\beta)(1-\beta q)}{(1-q)(1-q^2)}x^2 - \frac{(1-\beta^2)}{(1-q^2)} $$

$$ C_3(x;\beta|q) = \frac{8(1-\beta)(1-\beta q)(1-\beta q^2)}{(1-q)(1-q^2)(1-q^3)}x^3 - \frac{2x(1-\beta)[2+q+\beta(1-q^2)-\beta^2 q(1+2q)]}{(1-q^2)(1-q^3)} . $$

From this point on the formulas become more complicated.

Another useful expression for $C_n(\cos\theta;\beta|q)$ can be obtained by applying the q-analogue of the Pfaff-Kummer transformation to (3.2). This transformation is

(3.25)
$$ {}_2\varphi_1\left(\begin{matrix} a,\ b \\ c \end{matrix} ; q,x\right) = \frac{(ax;q)_\infty}{(x;q)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{c}{b};q\right)_n (a;q)_n (-xb)^n q^{\binom{n}{2}}}{(c;q)_n (q;q)_n (ax;q)_n} . $$

See Andrews [4] for a proof of this formula. The resulting formula is

$$(3.26) \qquad C_n(\cos\theta;\beta|q) = \frac{(\beta;q)_n}{(q;q)_n} e^{in\theta} (\beta^{-1}e^{-2i\theta}q^{1-n};q)_n \cdot$$

$$\cdot \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{1-n}\beta^{-2};q)_k q^{\binom{k}{2}} (-qe^{-2i\theta})^k}{(q^{1-n}\beta^{-1};q)_k (q^{1-n}\beta^{-1}e^{-2i\theta};q)_k (q;q)_k} .$$

Now sum this series in the opposite direction, i.e. replace k by $n - k$, and simplify to obtain

$$(3.27) \qquad C_n(\cos\theta;\beta|q) = \frac{(\beta^2;q)_n e^{-in\theta}}{(q;q)_n \beta^n} \; {}_3\varphi_2\left(\begin{matrix} q^{-n}, \beta, \beta e^{2i\theta} \\ \beta^2, 0 \end{matrix} ; q, q\right) .$$

From this it is easy to obtain the generating function

$$(3.28) \qquad \sum_{n=0}^{\infty} \frac{C_n(\cos\theta;\beta|q)}{(\beta^2;q)_n} q^{\binom{n}{2}} (\beta r)^n = (-re^{-i\theta};q)_\infty \; {}_2\varphi_1\left(\begin{matrix} \beta, \beta e^{2i\theta} \\ \beta^2 \end{matrix} ; q, -re^{-i\theta}\right) .$$

4. <u>The orthogonality relation</u>. The orthogonality relation for the q-Wilson polynomials defined in (3.3) is

$$(4.1)\qquad \int_{-1}^{1} \frac{p_n(x)p_m(x) \prod_{k=0}^{\infty} (1 - 2(2x^2 - 1)q^k + q^{2k})}{h(x,a)h(x,b)h(x,c)h(x,d)} \frac{dx}{\sqrt{1 - x^2}}$$

$$= \begin{cases} 0 & m \neq n \\ \dfrac{2\pi a^{2n}(bc;q)_n(bd;q)_n(cd;q)_n(q;q)_n(1 - abcdq^{n-1})}{(ab;q)_n(ac;q)_n(ad;q)_n(abcd;q)_n(1 - abcdq^{2n-1})} \cdot & m = n \\ \cdot \dfrac{(abcd;q)_\infty}{(ab;q)_\infty(ac;q)_\infty(ad;q)_\infty(bc;q)_\infty(bd;q)_\infty(cd;q)_\infty(q;q)_\infty} & \end{cases}$$

when $|a|,|b|,|c|,|d| < 1$, where

$$(4.2)\qquad h(x,a) = \prod_{n=0}^{\infty} (1 - 2axq^n + a^2q^{2n}) .$$

In general there will be finitely many mass points outside $(-1,1)$. See [12] for details.

For the continuous q-ultraspherical polynomials when $|\beta| < 1$, the specializations of the last section lead to

$$(4.3)\qquad \int_{-1}^{1} C_n(x;\beta|q)C_m(x;\beta|q)w_\beta(x)(1 - x^2)^{-1/2}dx = 2\pi \frac{(1 - \beta)}{(1 - \beta q^n)} \frac{(\beta^2;q)_n}{(q;q)_n} \frac{(\beta;q)_\infty(\beta q;q)_\infty}{(\beta^2;q)_\infty(q;q)_\infty} \delta_{m,n}$$

where

$$(4.4)\qquad w_\beta(x) = \prod_{k=0}^{\infty} \frac{(1 - 2(2x^2 - 1)q^k + q^{2k})}{(1 - 2(2x^2 - 1)q^k\beta + q^{2k}\beta^2)} .$$

This can also be written as

$$(4.5)\qquad w_\beta(\cos\theta) = \frac{(e^{2i\theta};q)_\infty(e^{-2i\theta};q)_\infty}{(\beta e^{2i\theta};q)_\infty(\beta e^{-2i\theta};q)_\infty} .$$

To get a better idea of this weight function observe that

$$w_{q^2}(x)(1 - x^2)^{-1/2} = (4 - 4x^2)((1 + q)^2 - 4qx^2)(1 - x^2)^{-1/2} .$$

When $q = 1$ this becomes $4^2(1 - x)^2)^{2-1/2}$.

Before considering other special cases we will give a direct proof of the orthogonality relation (4.3). There are two reasons for this. One is that no other proof has appeared yet, and the three other proofs that we know use results that are not needed in the present proof. Also this proof will give a direct proof of an important identity of Rogers [36].

As a first step we compute a trigonometric moment that is of independent interest,

$$\int_0^\pi e^{2ik\theta} w_\beta(\cos\theta)\,d\theta = \int_0^\pi e^{2ik\theta} \prod_{n=0}^\infty \frac{(1-r^{2i\theta}q^n)(1-e^{-2i\theta}q^n)}{(1-e^{2i\theta}\beta q^n)(1-e^{-2i\theta}\beta q^n)}\,d\theta .$$

Use the q-binomial theorem,

$$\frac{(e^{2i\theta};q)_\infty}{(\beta e^{2i\theta};q)_\infty} = \sum_{n=0}^\infty \frac{(\beta^{-1};q)_n}{(q;q)_n}\beta^n e^{2in\theta} ,$$

to get

$$\int_0^\pi e^{2ik\theta} w_\beta(\cos\theta)\,d\theta = \sum_{n=0}^\infty \frac{(\beta^{-1};q)_n}{(q;q)_n}\beta^n \sum_{m=0}^\infty \frac{(\beta^{-1};q)_m}{(q;q)_m}\beta^m \int_0^\pi e^{2i(k+n-m)\theta}d\theta$$

$$= \pi \sum_{n=0}^\infty \frac{(\beta^{-1};q)_n(\beta^{-1};q)_{k+n}}{(q;q)_n(q;q)_{k+n}}\beta^{k+2n}$$

$$= \pi\beta^k \frac{(\beta^{-1};q)_k}{(q;q)_k}\ {}_2\varphi_1\left(\begin{matrix}\beta^{-1},\ q^k\beta^{-1} \\ q^{k+1}\end{matrix};\ q,\beta^2\right) .$$

Heine [28], [29, p. 106], see also [5, Cor. 2.2], gave an important transformation of the general ${}_2\varphi_1$,

$$(4.6) \qquad {}_2\varphi_1\left(\begin{matrix}a,\ b \\ c\end{matrix};\ q,x\right) = \frac{(ax;q)_\infty(b;q)_\infty}{(x;q)_\infty(c;q)_\infty}\ {}_2\varphi_1\left(\begin{matrix}\frac{c}{b},\ x \\ ax\end{matrix};\ q,b\right) .$$

The iterate of this is

$$(4.7) \qquad {}_2\varphi_1\left(\begin{matrix}a,\ b \\ c\end{matrix};\ q,x\right) = \frac{(bx;q)_\infty(\frac{c}{b};q)_\infty}{(x;q)_\infty(c;q)_\infty}\ {}_2\varphi_1\left(\begin{matrix}\frac{abx}{c},\ b \\ bx\end{matrix};\ q,\frac{c}{b}\right) .$$

Use (4.7) on the ${}_2\varphi_1$ above to get

$$\int_0^\pi e^{2ik\theta} w_\beta(\cos\theta)d\theta = \frac{\pi\beta^k(\beta^{-1};q)_k(\beta;q)_\infty(\beta q^{k+1};q)_\infty}{(q;q)_k(\beta^2;q)_\infty(q^{k+1};q)_\infty}\ {}_2\varphi_1\left(\begin{matrix} q^{-1},\ \beta^{-1} \\ \beta \end{matrix};\ q,\beta q^{k+1}\right).$$

Summing this ${}_2\varphi_1$ (there are only two nonzero terms) gives

$$\int_0^\pi e^{2ik\theta} w_\beta(\cos\theta)d\theta = \frac{\pi\beta^k(\beta^{-1};q)_k(1+q^k)}{(\beta q;q)_k}\ \frac{(\beta;q)_\infty(\beta q;q)_\infty}{(q;q)_\infty(\beta^2,q)_\infty}, \tag{4.8}$$

$|\beta| < 1$. The condition $|\beta| < 1$ was used to expand $w_\beta(\cos\theta)$ by the q-binomial theorem, and the series that was used does not converge when $\beta^2 \geqq 1$. The case $\beta = 1$ is trivial, since $w_1(\cos\theta) = 1$. In this case

$$\int_0^\pi e^{2ik\theta} w_1(\cos\theta)d\theta = \begin{cases} 0, & k = \pm1,\pm2,\ldots, \\ \pi, & k = 0. \end{cases}$$

To continue the proof of the orthogonality relation consider

$$\int_0^\pi C_n(\cos\theta;\beta|q)T_{n-2k}(\cos\theta)w_\beta(\cos\theta)d\theta$$

where

$$T_n(\cos\theta) = \cos n\theta .$$

This integral is

$$\sum_{j=0}^n \frac{(\beta;q)_j(\beta;q)_{n-j}}{(q;q)_j(q;q)_{n-j}} \int_0^\pi e^{i(n-2j)\theta}\left[\frac{e^{i(n-2k)\theta} + e^{-i(n-2k)\theta}}{2}\right] w_\beta(\cos\theta)d\theta$$

$$= \frac{1}{2}\sum_{j=0}^n \frac{(\beta;q)_j(\beta;q)_{n-j}}{(q;q)_j(q;q)_{n-j}} \int_0^\pi [e^{2i(n-j-k)\theta} + e^{2i(k-j)\theta}]w_\beta(\cos\theta)d\theta$$

$$= \frac{\pi}{2}\ \frac{(\beta;q)_\infty(\beta q;q)_\infty}{(\beta^2;q)_\infty(q;q)_\infty}\ [I_{n-k} + I_k]$$

where

$$I_k = \sum_{j=0}^n \frac{(\beta;q)_j(\beta;q)_{n-j}}{(q;q)_j(q;q)_{n-j}}\ \frac{\beta^{k-j}(\beta^{-1};q)_{k-j}}{(\beta q;q)_{k-j}}(1+q^{k-j}) .$$

Use

$$\frac{(a;q)_{n-j}}{(b;q)_{n-j}} = \frac{(a;q)_n}{(b;q)_n} \frac{(q^{1-n}b^{-1};q)_j}{(q^{1-n}a^{-1};q)_j} \left(\frac{b}{a}\right)^j$$

to rewrite this as

$$(4.9) \qquad I_k = \frac{(\beta;q)_n(\beta^{-1};q)_k}{(q;q)_n(\beta q;q)_k} \beta^k(1+q^k) \sum_{j=0}^{n} \frac{(q^{-n};q)_j(q^{-k}\beta^{-1};q)_j(\beta;q)_j(-q^{1-k};q)_j}{(q;q)_j(q^{1-n}\beta^{-1};q)_j(q^{1-k}\beta;q)_j(-q^{-k};q)_j} q^j .$$

This sum is a balanced ${}_4\varphi_3$ (i.e. the product of the numerator parameters times q is the product of the denominator parameters that are listed in the ${}_4\varphi_3$) and so may be transformed by

$$(4.10) \qquad {}_4\varphi_3\left(\begin{matrix} q^{-n},\ a,\ b,\ c \\ d,\ e,\ f \end{matrix} ;\ q,q\right) = \left(\frac{bc}{d}\right)^n \frac{\left(\frac{de}{bc};q\right)_n\left(\frac{df}{bc};q\right)_n}{(e;q)_n(f;q)_n}\ {}_4\varphi_3\left(\begin{matrix} q^{-n},\ a,\ \frac{d}{c},\frac{d}{b} \\ d,\ \frac{de}{bc},\frac{df}{bc} \end{matrix} ;\ q,q\right)$$

where $q^{1-n}abc = def$. See [7] or [12] for this transformation. Let $d = -q^{-k}$, $a = \beta$. This gives

$$(4.11) \qquad I_k = \frac{(\beta;q)_n(\beta^{-1};q)_k\beta^{k-n}(1+q^k)q^{(1-k)n}(q^{k-n};q)_n(\beta^2;q)_n}{(q;q)_n(\beta q;q)_k(q^{1-n}\beta^{-1};q)_n(q^{1-k}\beta;q)_n} .$$

$$\cdot\ {}_4\varphi_3\left(\begin{matrix} q^{-n},\ \beta,\ q^{-1},\ -\beta \\ -q^{-k},\ q^{k-n},\ \beta^2 \end{matrix} ;\ q,q\right)$$

$$= (\beta q)^k \frac{(1-q^{n-2k})(\beta^{-1};q)_k(\beta^2;q)_n(q^{1-k};q)_{n-1}}{(1-q^{1-k}\beta)(\beta q;q)_k(q;q)_n(q^{2-k}\beta;q)_{n-1}} .$$

Thus $I_k = 0$, $k = 1,2,\ldots,n-1$, and $I_0 = I_n = (\beta^2;q)_n/(\beta q;q)_n$. Since

$$T_n(x) = 2^{n-1}x^n + \cdots, \quad n = 1,2,\ldots, \quad T_0(x) = 1 ,$$

and

$$C_n(x;\beta|q) = 2^n \frac{(\beta;q)_n}{(q;q)_n} x^n + \cdots, \qquad n = 0,1,\ldots ,$$

it is easy to check that (4.3) holds.

The argument above is easy when $|\beta| < 1$, it would be more complicated for the values of β with $|\beta| > 1$ when the polynomials are orthogonal with respect to a positive measure. Rather than try to carry out this argument (we have not tried)or introduce the methods that we know will lead to this orthogonality we will now use the orthogonality to find some important identities. The complete orthogonality relations will be given in [11] and [12]. Different methods are used in these two papers.

The argument above did not need $k = 0,1,\ldots,n$. Let k be successively $-k$ and $n + k$ in (4.11). A calculation gives

$$\int_0^\pi C_n(\cos\theta;\beta|q)T_{n+2k}(\cos\theta)w_\beta(\cos\theta)d\theta \tag{4.12}$$

$$= \frac{\pi\beta^k(\beta^{-1};q)_k(q;q)_{n+k}}{(q;q)_k(q;q)_n}\frac{(1-q^{n+2k})}{(1-q^{n+k})}\frac{(\beta;q)_\infty(\beta q^{n+k+1};q)_\infty}{(q;q)_\infty(\beta^2q^n;q)_\infty}, \quad k = 1,2,\ldots .$$

Using the orthogonality relation (4.3) and (4.12) gives

$$T_n(x) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\beta^k\frac{(\beta^{-1};q)_k(q;q)_{n-k}(1-q^n)(1-\beta q^{n-2k})}{(q;q)_k(\beta q;q)_{n-k}(1-q^{n-k})2(1-\beta)}C_{n-2k}(x;\beta|q) . \tag{4.13}$$

Then if (4.13) is used on the right hand side of (3.11), the result is

$$C_n(x;\gamma|q) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(\gamma;q)_k(\gamma;q)_{n-k}(q;q)_{n-2k}}{(q;q)_k(q;q)_{n-k}(\beta q;q)_{n-2k}}(1-\beta q^{n-2k}) \cdot \tag{4.14}$$

$$\cdot\ {}_6\varphi_5\left\{\begin{matrix} q^{n-2k},\ q^{n/2-k+1},\ -q^{n/2-k+1},\ \gamma q^{n-k},\ \beta^{-1},\ q^{-k} \\ q^{n/2-k},\ -q^{n/2-k},\ \gamma^{-1}q^{1-k},\ \beta q^{n+1-2k},\ q^{n+1-k}\end{matrix};\ q,\ \frac{\beta q}{\gamma}\right\}\frac{C_{n-2k}(x;\beta|q)}{(1-\beta)} .$$

The ${}_6\varphi_5$ is very well poised and so can be summed by a theorem of Jackson [37, p. 96]. The resulting series is

$$C_n(x;\beta|q) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\beta^k\frac{(\gamma\beta^{-1};q)_k(\gamma;q)_{n-k}}{(q;q)_k(\beta q;q)_{n-k}}\frac{(1-\beta q^{n-2k})}{(1-\beta)}C_{n-2k}(x;\beta|q) . \tag{4.15}$$

This very important formula was found by Rogers [36]. He obtained it by finding the coefficients for small values of n, guessing the answer, and then proving it by

induction. The special case when $\gamma = 0$ and $\beta = 1$ was used in his second paper [35] to obtain the identities

$$(4.16)\qquad \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

and

$$(4.17)\qquad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}\ .$$

While it is not particularly easy to read and understand these papers of Rogers, we have found it easier to read them after understanding the polynomials of Rogers. It is also easier to read them in the order [36], [35], [34] rather than the order in which they were written. These papers are very interesting and contain many other important results. Probably the most interesting is

$$(4.18)\qquad C_m(x;\beta|q)C_n(x;\beta|q) = \sum_{k=0}^{\min(m,n)} a(k,m,n)C_{m+n-2k}(x;\beta|q)$$

$$a(k,m,n) = \frac{(q;q)_{m+n-2k}(\beta;q)_{n-k}(\beta;q)_{m-k}(\beta;q)_k(\beta^2;q)_{m+n-k}(1-\beta q^{m+n-2k})}{(\beta^2;q)_{m+n-2k}(q;q)_{n-k}(q;q)_{m-k}(q;q)_k(\beta q;q)_{m+n-k}(1-\beta)}\ .$$

Rogers found this result in the same way he found (4.15), by working out the coefficients for small m, guessing the answer, and proving it by induction. We do not have a better proof of (4.18) at this time.

Rogers [36] pointed out that the special case $q = 1$, $\beta = q^\lambda$ is an important result for spherical harmonics. However no one picked up this result from [36] and the special case $\beta = q^\lambda$, $q = 1$ of (4.18) was next stated by Dougall [19] almost twenty-five years later. The only special cases that were known before Rogers found the general result were the trivial cases $\beta = 1$, which is equivalent to

$$\cos n\theta \cos m\theta = \frac{1}{2}[\cos(n+m)\theta + \cos(n-m)\theta]\ ,$$

$\beta = q$, which is

$$\frac{\sin(n+1)\theta}{\sin\theta}\,\frac{\sin(m+1)\theta}{\sin\theta} = \sum_{k=0}^{\min(m,n)} \frac{\sin(m+n+1-2k)\theta}{\sin\theta},$$

and one nontrivial case, $\beta = q^{1/2}$, $q = 1$,

$$P_n(x)P_m(x) = \sum_{k=0}^{\min(m,n)} \frac{m+n+\frac{1}{2}-2k}{m+n+\frac{1}{2}-k}\,\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_{m-k}\left(\frac{1}{2}\right)_{n-k}(m+n-k)!}{k!(m-k)!(n-k)!\left(\frac{1}{2}\right)_{m+n-k}}\,P_{m+n-2k}(x),$$

a result of Ferrers [24, Example 10, p. 156] and Adams [1]. See Chapter 5 of [9] for a summary of some of the known results on the linearization problem.

Formulas (4.15) and (4.18) can be inverted. The inverse of (4.18) is

$$(4.19)\qquad \frac{(\beta;q)_m(\beta;q)_n}{(q;q)_m(q;q)_n}C_{m+n}(x;\beta|q) = \frac{(\beta;q)_{m+n}}{(q;q)_{m+n}}\sum_{k=0}^{\min(m,n)} b(k,m,n)C_{m-k}(x;\beta|q)C_{n-k}(x;\beta|q),$$

$$b(k,m,n) = \frac{(q^{-m-n}\beta^{-2};q)_k(1-q^{2k-m-n}\beta^{-2})(\beta^{-1};q)_k}{(q;q)_k(1-q^{-m-n}\beta^{-2})(q^{1-m-n}\beta^{-1};q)_k}\left(\frac{\beta^2}{q}\right)^k.$$

The limiting case $\beta = q^\lambda$, $q = 1$ of (4.19) was found by Al-Salam [3]. A proof of (4.19) can be given following his proof; use (4.18) to replace $C_{m-k}(x;\beta|q)C_{n-k}(x;\beta|q)$ and invert the order of summation. The resulting series is a very well poised ${}_6\varphi_5$, which is summed by Jackson's result. Details are left to the reader.

The other inverse is

$$(4.20)\qquad w_\beta(x)C_n(x;\beta|q) = \sum_{k=0}^{\infty} a(k,n)C_{n+2k}(x;\gamma|q)w_\gamma(x),$$

$$a(k,n) = \frac{\beta^k(\gamma/\beta;q)_k(q^{n+1};q)_{2k}(\gamma^2q^{n+2k};q)_\infty(\beta q^{n+k+1};q)_\infty(\beta;q)_\infty(1-\gamma q^{n+2k})}{(q;q)_k(\gamma q^{n+k};q)_\infty(\beta^2q^n;q)_\infty(\gamma;q)_\infty}.$$

This follows from (4.15) by the general argument given by Askey in [8]. The series (4.20) converges when $|\beta| < 1$, $|q| < 1$. For $|a(k,n)| = O(\beta^k)$ and

$$|C_{n+2k}(x;\gamma|q)| \leq (n+1)\max_k\left|\frac{(\gamma;q)_k}{(q;q)_k}\right| = A(\gamma,q)(n+1)$$

for γ, q fixed, $|q| < 1$.

Rogers [36] found a number of other formulas. The reader is referred to [36] for these results.

5. Special cases. It is not surprising that interesting results are found when $q \to 1$. However it is surprising that an interesting result could be found by letting $q \to 0$. To see this consider the generating function (2.10) when $q = 0$. It is

$$(5.1) \qquad \frac{1 - 2\beta r \cos\theta + \beta^2 r^2}{1 - 2r\cos\theta + r^2} = \frac{(1 - \beta r e^{i\theta})(1 - \beta r e^{-i\theta})}{(1 - re^{i\theta})(1 - re^{-i\theta})} = \sum_{n=0}^{\infty} C_n(\cos\theta; \beta, 0) r^n .$$

A partial fraction decomposition gives

$$(5.2) \qquad C_n(\cos\theta;\beta|0) = 2\beta(1-\beta)\cos n\theta + (1-\beta)^2 \frac{\sin(n+1)\theta}{\sin\theta}, \quad n = 1,2,\dots,$$

$$C_0(\cos\theta;\beta|0) = 1$$

or

$$C_n(x;\beta|0) = 2\beta(1-\beta)T_n(x) + (1-\beta)^2 U_n(x), \qquad n = 1,2,\dots,$$

$$C_0(x;\beta|0) = 1 .$$

These can also be written as

$$C_n(\cos\theta;\beta|0) = (1-\beta)\frac{\sin(n+1)\theta}{\sin\theta} - \beta(1-\beta)\frac{\sin(n-1)\theta}{\sin\theta}, \qquad n = 1,2,\dots,$$

$$C_0(\cos\theta;\beta|0) = 1$$

or

$$C_n(x;\beta|0) = (1-\beta)U_n(x) - \beta(1-\beta)U_{n-2}(x), \qquad n = 1,2,3,\dots,$$

$$C_0(x;\beta|0) = 1 .$$

The orthogonality relation is

$$(5.3) \qquad \int_{-1}^{1} C_n(x;\beta|0)C_m(x;\beta|0)\frac{(1-x^2)^{1/2}}{(1+\beta)^2 - 4\beta x^2}\,dx = \frac{\pi}{2}(1-\beta)^2\delta_{m,n}, \quad (m,n) \neq (0,0)$$

$$\int_{-1}^{1} [C_0(x;\beta|0)]^2 \frac{(1-x^2)^{1/2}}{(1+\beta)^2 - 4\beta x^2}\,dx = \frac{\pi}{2(1+\beta)}$$

when $|\beta| < 1$.

This is a known result. It follows easily from the Szegö-Bernstein theory [14], [40, §2.6]; it was also given by Karlin and McGregor [31]. When $\beta > 1$ there are two point masses that need to be added outside $[-1,1]$. See Karlin and McGregor [31].

Some of these functions are spherical functions on rank one spaces over reductive p-adic groups. See Cartier [18].

The case $q \to 1$ gives the classical ultraspherical polynomials. The main point of interest here is the way the weight functions converge as $q \to 1$. Recall that

$$\int_{-1}^{1} \prod_{n=0}^{\infty} \frac{(1 - 2(2x^2 - 1)q^n + q^{2n})}{(1 - 2(2x^2 - 1)\beta q^n + \beta^2 q^{2n})} \frac{dx}{\sqrt{1 - x^2}} = 2\pi \frac{(\beta;q)_\infty (\beta q;q)_\infty}{(q;q)_\infty (\beta^2;q)_\infty} . \tag{5.4}$$

Let $\beta = q^\lambda$ and $x = e^{i\theta}$. Then (5.4) becomes

$$\int_{0}^{\pi} \prod_{n=0}^{\infty} \frac{(1 - e^{2i\theta}q^n)}{(1 - e^{2i\theta}q^{n+\lambda})} \prod_{n=0}^{\infty} \frac{(1 - e^{-2i\theta}q^n)}{(1 - e^{-2i\theta}q^{n+\lambda})} d\theta = 2\pi \frac{(q^\lambda;q)_\infty (q^{\lambda+1};q)_\infty}{(q^{2\lambda};q)_\infty (q;q)_\infty} . \tag{5.5}$$

The q-gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1 . \tag{5.6}$$

In [10] it was shown that $\lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x)$. The limit on the right hand side of (5.5) can be written as

$$\frac{2\pi\Gamma_q(2\lambda)}{\Gamma_q(\lambda)\Gamma_q(\lambda + 1)} \to \frac{2\pi\Gamma(2\lambda)}{\Gamma(\lambda)\Gamma(\lambda + 1)} \quad \text{as } q \to 1 . \tag{5.7}$$

The left hand side of (5.5) is

$$\int_0^\pi {}_1\varphi_0(\underline{q^{-\lambda}}; q, q^\lambda e^{2i\theta}) \, {}_1\varphi_0(\underline{q^{-\lambda}}; q, q^\lambda e^{-2i\theta}) d\theta ,$$

and formally this converges to

$$\int_0^\pi (1 - e^{2i\theta})^\lambda (1 - e^{-2i\theta})^\lambda d\theta = \int_0^\pi (2 - 2\cos\theta)^\lambda d\theta = 2^{2\lambda} \int_0^\pi (\sin\theta)^{2\lambda} d\theta .$$

The formal argument can be justified since

$${}_1\varphi_0(\underline{a}; q, x) = \frac{(ax;q)_\infty}{(x;q)_\infty}$$

and

$${}_1F_0(\underline{a}; x) = (1 - x)^{-a}$$

are functions which are analytic for $|x| < 1$ and have extensions to the complex plane

cut along $(-\infty,-1]$ and

$$\lim_{q\to 1} {}_1\varphi_0(q^{\underline{a}};\, q,x) = {}_1F_0(\underline{a};\, x) \ .$$

Another interesting case is $q \to -1$. Consider the case $\beta = -|q|^{\lambda}$. Set $q = -p$.

Then

$$W_{-|q|^{\lambda},q}(\cos\theta) = \left|\frac{(e^{2i\theta};q)_\infty}{(-|q|^{\lambda}e^{2i\theta};q)_\infty}\right|^2 = \left|\frac{(e^{2i\theta};q^2)_\infty(qe^{2i\theta};q^2)_\infty}{(-|q|^{\lambda}e^{2i\theta};q^2)_\infty(-q|q|^{\lambda}e^{2i\theta};q^2)_\infty}\right|^2$$

$$= \left|\frac{(e^{2i\theta};p^2)_\infty(-pe^{2i\theta};p^2)_\infty}{(p^{\lambda+1}e^{2i\theta};p^2)_\infty(-p^{\lambda}e^{2i\theta};p^2)_\infty}\right|^2$$

$$= \left|{}_1\varphi_0(p_{-}^{-(\lambda+1)};\, p^2,\, p^{\lambda+1}e^{2i\theta})\ {}_1\varphi_0(-p_{-}^{-(\lambda-1)};\, p^2,\, -p^{\lambda}e^{2i\theta})\right|^2 .$$

As $p \to 1$ this converges to

$$\left|{}_1F_0\left(\begin{matrix}-(\lambda+1)/2\\ -\end{matrix};\, e^{2i\theta}\right){}_1F_0\left(\begin{matrix}(1-\lambda)/2\\ -\end{matrix};\, -e^{2i\theta}\right)\right|^2 = \left|(1-e^{2i\theta})^{(\lambda+1)/2}(1+e^{2i\theta})^{(\lambda-1)/2}\right|^2$$

$$= (2-2\cos 2\theta)^{(\lambda+1)/2}(2+2\cos 2\theta)^{(\lambda-1)/2}$$

$$= 2^{\lambda+1}(\sin^2\theta)^{(\lambda+1)/2}2^{\lambda-1}(\cos^2\theta)^{(\lambda-1)/2}$$

$$= 2^{2\lambda}(1-x^2)^{(\lambda+1)/2}|x|^{\lambda-1} \ .$$

Set

(5.8) $$N_n^{\lambda}(\cos\theta) = \lim_{q\to -1} C_n(\cos\theta;\, -|q|^{\lambda}|q) \ .$$

Then

(5.9) $$\int_{-1}^{1} N_m^{\lambda}(x)N_n^{\lambda}(x)(1-x^2)^{\lambda/2}|x|^{\lambda-1}dx = 0, \quad m \neq n \ .$$

A change of variables gives

(5.10) $$N_{2m}^{\lambda}(x) = a_m P_m^{(\frac{\lambda}{2},\frac{\lambda}{2}-1)}(2x^2-1) \ ,$$

(5.11) $$N_{2m+1}^{\lambda}(x) = b_m x P_m^{(\frac{\lambda}{2},\frac{\lambda}{2})}(2x^2-1) \ ,$$

where the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is defined by

(5.12) $$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+1\\ \alpha+1\end{matrix};\ \frac{1-x}{2}\right).$$

They are orthogonal on $(-1,1)$ with respect to $(1-x)^\alpha(1+x)^\beta$. The constants a_m and b_m in (5.10) and (5.11) can be determined by keeping track of the constants that occur in the various orthogonality relations, they can be determined from the recurrence relations or the coefficient of x^n in one of the explicit formulas for $C_n(x;\beta|q)$ can be computed under the specialization $\beta = -|q|^\lambda$ and $q \to -1$, and can be compared with the coefficient of x^n in (5.12). The last is the easiest method. The results are

(5.13) $$N_{2m}^\lambda(x) = \frac{(\lambda)_m}{(\frac{\lambda}{2})_m} P_m^{(\frac{\lambda}{2},\frac{\lambda}{2}-1)}(2x^2-1)$$

and

(5.14) $$N_{2m+1}^\lambda(x) = 2x\frac{(\lambda+1)_m}{(\frac{\lambda+1}{2})_m} P_m^{(\frac{\lambda}{2},\frac{\lambda}{2})}(2x^2-1).$$

Now we can see the real reason for the existence of a surprising result of Hylleraas [30]. The coefficients in the expansion

(5.15) $$P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n)P_k^{(\alpha,\beta)}(x)$$

are of interest. The case $\alpha = \beta$ is the special case $\beta = q^{\alpha+1/2}$, $q \to 1$ of (4.18). The coefficients in (5.15) are not usually given as a single term, which they are when $\alpha = \beta$. They are usually a sum of products, rather than just a product. Hylleraas found one other case when the coefficients are given by a product, the case $\alpha = \beta + 1$. This follows from (4.18) when n and m are even, $\beta = -|q|^\lambda$ and $q \to -1$.

The case $\beta = |q|^\lambda$, $q \to -1$ leads to similar results, but nothing new, so it will not be considered here.

6. The continuous q-Hermite polynomials. The last special case that will be mentioned is the case $\beta = 0$. With a different normalization they will be called the continuous q-Hermite polynomials,

(6.1) $$H_n(x|q) = (q,q)_n C_n(x;0|q) .$$

The recurrence relation becomes

(6.2) $$2xH_n(x|q) = H_{n+1}(x|q) + (1 - q^n)H_{n-1}(x|q) .$$

The generating function (2.10) becomes

(6.3) $$\frac{1}{|(re^{i\theta};q)_\infty|^2} = \sum_{n=0}^{\infty} \frac{H_n(\cos\theta|q)r^n}{(q;q)_n}$$

or

(6.4) $$\frac{1}{\prod_{n=0}^{\infty}(1 - 2xrq^n + r^2q^{2n})} = \sum_{n=0}^{\infty} \frac{H_n(x|q)r^n}{(q;q)_n} .$$

Rogers [35] and later Szegö [39] and Carlitz [15], [16] studied these polynomials extensively. Carlitz used a different variable and normalization, so his results need a slight translation to correspond to those of Rogers and special cases of formulas in this paper. He uses a variable and normalization that make it clear that the functions are polynomials in this variable. However the price he has to pay of losing the nice orthogonality relation is so high that we will use the notation above.

The q-extension of Mehler's bilateral sum for Hermite polynomials is

(6.5) $$\frac{(r^2;q)_\infty}{|(re^{i(\theta+\varphi)};q)_\infty(re^{i(\theta-\varphi)};q)_\infty|^2} = \sum_{n=0}^{\infty} \frac{H_n(\cos\theta|q)H_n(\cos\varphi|q)r^n}{(q;q)_n} .$$

See [40, Problem 23] for Mehler's formula. A beautiful combinatorial proof of Mehler's formula has been given by Foata [25]. Formula (6.5) is one of the few results that have been found for q-Hermite polynomials that has not been extended to q-ultraspherical polynomials.

The orthogonality relation (4.3) becomes

$$(6.6)\quad \int_{-1}^{1} H_n(x|q)H_m(x|q) \prod_{k=0}^{\infty} (1 - 2(2x^2 - 1)q^k + q^{2k})(1 - x^2)^{-1/2}dx = \frac{2\pi(q;q)_n}{(q;q)_\infty}\,\delta_{m,n}\ ,$$

so (6.5) is the analogue of Mehler's formula. For they both have the form

$$\sum_{n=0}^{\infty} \frac{p_n(x)p_n(n)r^n}{h_n}$$

with

$$h_n = \int_a^b [p_n(x)]^2 w(x)dx\ .$$

The orthogonality relation (6.6) was given by Allaway [2].

The reader should now read the papers of Rogers [34], [35], and [36] to see how he used the q-Hermite polynomials to obtain the Rogers-Ramanujan identities and many other results.

There are many open problems. The most important is to find spaces on which these polynomials live, presumably as spherical functions. Then groups acting on these spaces need to be used to derive addition formulas. It would be interesting to find a combinatorial interpretation of the q-Hermite polynomials, and then use it to give a combinatorial proof of (6.5). Slepian [38] has a very important multisum extension of Mehler's formula and Foata and Garsia [26] have found a combinatorial proof of this formula. A similar formula for the continuous q-Hermite polynomials should be obtained.

REFERENCES

[1] J. C. Adams, On the expression of the product of any two Legendre's coefficients by means of a series of Legendre's coefficients, Proc. Royal Soc. London, 27 (1878), 63-71.

[2] W. Allaway, The identification of a class of orthogonal polynomial sets, Ph.D. thesis, U. of Alberta, Edmonton, Alberta, Canada, 1972.

[3] W. A. Al-Salam, On the product of two Legendre polynomials, Math. Scand., 4 (1956), 239-242.

[4] G. Andrews, On the q-analog of Kummer's theorem and applications, Duke Math. J., 40 (1973), 525-528.

[5] G. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and Its Applications, Vol. 2, Addison-Wesley, Reading, MA., 1977.

[6] G. Andrews and R. Askey, Enumeration of partitions: The role of Eulerian series and q-orthogonal polynomials, Higher Combinatorics, ed. M. Aigner, D. Reidel, Dordrecht, Boston, 1977, 3-26.

[7] G. Andrews and R. Askey, q-analogues of the classical orthogonal polynomials and applications, to appear.

[8] R. Askey, Orthogonal expansions with positive coefficients, Proc. Amer. Math. Soc., 16 (1965), 1191-1194.

[9] R. Askey, Orthogonal Polynomials and Special Functions, Regional Conference Series in Applied Mathematics 21, SIAM, Philadelphia, 1975.

[10] R. Askey, The q-gamma and q-beta functions, Applicable Analysis, to appear.

[11] R. Askey and M. Ismail, Recurrence relations, continued fractions and orthogonal polynomials, to appear.

[12] R. Askey and J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, to appear.

[13] W. N. Bailey, Generalized Hypergeometric Series, Cambridge Univ. Press, Cambridge, 1935.

[14] S. Bernstein, Sur les polynomes orthogonaux relatifs à un segment fini, J. de Math. (9) Vol. 9 (1930), 127-177; Vol. 10 (1931), 219-286.

[15] L. Carlitz, Some polynomials related to theta functions, Ann. Math. Pura Appl., Ser. 4, 41 (1955), 359-373.

[16] L. Carlitz, Some polynomials related to theta functions, Duke Math. J. 24 (1957), 521-527.

[17] L. Carlitz, Some formulas of F. H. Jackson, Monatshefte für Math., 73 (1969), 193-198.

[18] P. Cartier, Harmonic analysis on trees, Harmonic Analysis on Homogeneous Spaces, ed. C. C. Moore, Proc. Symp. Pure Math. 26 (1973), Amer. Math. Soc. Providence, R.I., 419-424.

[19] J. Dougall, A theorem of Sonine in Bessel functions, with two extensions to spherical harmonics, Proc. Edinburgh Math. Soc., 37 (1919), 33-47.

[20] J. Favard, Sur les polynômes de Tchebycheff, C. R. Acad. Sci. Paris, 200 (1935), 2052-2055.

[21] L. Fejér, Abschätzungen für die Legendreschen und verwandte Polynome, Math. Zeit., 24 (1925), 285-294, Gesammelte Arbeiten II, 161-175.

[22] L. Fejér, Potenzreihen mit mehrfach monotoner Koeffizientenfolge und ihre Legendre Polynome, Proc. Camb. Phil. Soc., 31 (1935), 307-316, Gesammelte Arbeiten II, 621-631.

[23] E. Feldheim, Sur les polynomes généralisés de Legendre, Izv. Akad. Nauk. SSSR Ser. Math., 5 (1941), 241-248, Russian Transl. ibid, 248-254.

[24] N. M. Ferrers, An Elementary Treatise on Spherical Harmonics and Subjects Connected with Them, Macmillan, London, 1877.

[25] D. Foata, A combinatorial proof of the Mehler formula, J. Comb. Th. (A), 24 (1978),

[26] D. Foata and A. M. Garsia, A combinatorial approach to Mehler formulas for Hermite polynomials, to appear.

[27] G. Freud, Orthogonale Polynome, Birkhäuser-Verlag, Basel and Stuttgart, 1969.

[28] E. Heine, Untersuchungen über die Reihe ..., J. Reine Angew. Math., 34 (1847), 285-328.

[29] E. Heine, Theorie der Kugelfunctionen, Berlin, 1878.

[30] E. Hylleraas, Linearization of products of Jacobi polynomials, Math. Scand., 10 (1962), 189-200.

[31] S. Karlin and J. McGregor, Random walks, Illinois J. Math., 3 (1959), 66-81.

[32] I. L. Lanzewizky, Über die Orthogonalität de Féjèr-Szegöschen Polynome, C. R. (Dokl.) Acad. Sci. URSS, 31 (1941), 199-200.

[33] P. Nevai, Orthogonal Polynomials, Memoirs of the Amer. Math. Soc.

[34] L. J. Rogers, On the expansion of some infinite products, Proc. London Math. Soc., 24 (1893), 337-352.

[35] L. J. Rogers, Second memoir on the expansion of certain infinite products, Proc. London Math. Soc., 25 (1894), 318-343.

[36] L. J. Rogers, Third memoir on the expansion of certain infinite products, Proc. London Math. Soc., 26 (1895), 15-32.

[37] L. J. Slater, Generalized Hypergeometric Functions, Cambridge Univ. Press, Cambridge, 1966.

[38] D. Slepian, On the symmetrized Kronecker power of a matrix and extensions of Mehler's formula for Hermite polynomials, SIAM J. Math. Anal., 3 (1972), 606-616.

[39] G. Szegö, Ein Beitrag zur Theorie der Thetafunktionen, Sitz. Preuss. Akad. Wiss. Phys. Math. Kl., XIX (1926), 242-252.

[40] G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Coll. Publ. XXIII, fourth edition, Providence, R.I., 1975.

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Some old polynomials of L. J. Rogers are orthogonal. Their weight function is given. The connection coefficient problem, which Rogers solved by guessing the formula and proving it by induction, is derived in a natural way and some other formulas are obtained. These polynomials generalize zonal spherical harmonics on spheres and include as special cases polynomials that are spherical functions on rank one spaces over reductive p-adic groups. A limiting case contains some Jacobi polynomials studied by Hylleraas that arose in work on the Yukawa potential.

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